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On the dispersion of small particles suspended in an isotropic turbulent fluid

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A solution to the dispersion of small particles suspended in a turbulent fluid is presented, based on the approximation proposed by Phythian for the dispersion of fluid points in an incompressible random fluid. Motion is considered in a frame moving with the mean velocity of the fluid, the forces acting on the particle being taken as gravity and a fluid drag assumed linear in the particle velocity relative to that of the fluid. The probability distribution of the fluid velocity field in this frame is taken as Gaussian, homogeneous, isotropic, stationary and of zero mean. It is shown that, in the absence of gravity, the long-time particle diffusion coefficient is in general greater than that of the fluid, approaching with increasing particle relaxation time a value consistent with the particle being in an Eulerian frame of reference. The effect of gravity is consistent with Yudine's effect of crossing trajectories, reducing unequally the particle diffusion in directions normal to and parallel to the direction of the gravitational field. To characterize the effect of flow and gravity on particle diffusion it has been found useful to use a Froude number defined in terms of the turbulent intensity rather than the mean velocity. Depending upon the value of this number, it is found that the particle integral time scale may initially decrease with increasing particle relaxation time though it eventually rises and approaches the particle relaxation time. It is finally shown how this analysis may be extended to include the extra forces generated by the fluid and particle accelerations.

1. Introduction

It is widely recognized that the deposition of particulate can present problems in the operation of large-scale systems, e.g. deposition from a nuclear-reactor coolant. An essential element in that process is the way in which particles are dispersed throughout the bulk of the fluid under the action of turbulent forces which are random in both space and time. Although such motion for 'large' particles is stochastically equivalent to Brownian motion and as such is well understood, no transport equation has yet been rigorously formulated to describe the collective motion of particles over the entire range of time scales of the particle–fluid interaction. For the most part gradient transport models based on Fick's law have been used in analogy with both Brownian motion and the transport of a passive scalar by a turbulent field. In the case of the latter the experimental and theoretical evidence as to the validity of such models is restricted to very simple types of turbulence (Batchelor 1952; Corrsin 1974). Although motion of a passive scalar is a special case of 'real' particle motion it illustrates some of the inherent problems in obtaining a collective description of their motion. For the most part transport of a passive scalar has been formulated in a Eulerian framework, where the problem reduces to one of closure of the statistical moment equations central to the transport mechanism. Most notable in this respect are Roberts' (1961) use of direct interaction and Saffman's (1969) and Phythian's (1972) approximations based on functional expansions in Gaussian random field variables. In each method, within the limit of the implied closure approximation gradient transport equations are derived for isotropic, homogeneous and stationary turbulence for times greater than the time scale of the turbulence.

None of these techniques, however, give an adequate description of 'real' particles with finite size and finite inertia since they fail to take account of the lack of coincidence between the particle and fluid-point trajectories. In view of the difficulty of this problem most authors have assumed the validity of Fick's law and have calculated the particle diffusion coefficient from the basic expression given by Taylor (1921) in his theory of 'diffusion by continuous movements', namely that for a stationary homogeneous field the time-varying particle diffusion coefficient $\epsilon_{ij}^{(p)}(\tau)$ is given by

$$\epsilon_{ij}^{(p)}(\tau) = \int_0^\tau \langle \Delta v_i(0)\, \Delta v_j(t)\rangle\, dt, \tag{1.1}$$

where $\Delta v_j(t)$ is the velocity of the particle in the j direction at time t relative to its mean and the angle brackets indicate an ensemble average over all realizations of the particle motion. In this respect we refer specifically to the work of Tchen (1947), Friedlander (1957), Peskin (1962), Csanady (1963), Hutchinson, Hewitt & Dukler (1971) and Meek & Jones (1973). Although Taylor's theory circumvents closure, as is well known, the relevant statistical correlations are those obtained along a particle trajectory for all realizations of the particle motion (Lagrangian) and the essential problem of nonlinearity is made manifest in finding a relationship between Lagrangian variables and those of the field at a stationary point (Eulerian). Tchen's analysis used an equation of motion which was consistent with a particle accelerating through a viscous time-varying fluid field in which the particle Reynolds number was small compared with unity (Hinze 1959). The significant feature of this analysis was that in the long-time limit equation (1.1) was identical to

$$\epsilon_{ij}^{(p)}(\infty) = \int_0^\infty \langle \Delta u_i(0)\, \Delta u_j(t)\rangle\, dt, \tag{1.2}$$

where $\Delta u_j(t)$ is the equivalent fluctuation in the fluid velocity relative to its mean at a point instantaneously occupied by the particle at time t for a particular realization of the particle's motion. Tchen, however, identified $\langle \Delta u_i(0)\, \Delta u_j(t)\rangle$ with the Lagrangian velocity correlation of the fluid itself and concluded that the particle and fluid diffusion coefficients were equal in the limit when both became time independent. This correlation function is clearly dependent upon particle motion, reflecting in general a dependence upon particle inertia, made manifest in an inability to follow the fluid oscillations, and also upon the effect of any external force acting upon the particle. For this reason, we shall hereafter refer to $\langle \Delta u_i(0)\, \Delta u_j(t)\rangle$ as $U_{ij}^{(p)}(t)$, where the superscript p symbolizes the dependence on particle motion. The effect of a constant external force was first recognized by Yudine (1959) and later used by Csanady (1963) in his analysis of the turbulent diffusion of heavy particles in the atmosphere. Here the particle inertia was considered sufficiently small for a particle to follow the fluid oscillations but the particle was considered sufficiently heavy for its gravitational drift significantly to affect its velocity correlation compared with that of the fluid. If $R_{ij}(\mathbf{x}, t)$ is the fluid Eulerian space–time velocity autocorrelation in a frame of reference moving with the fluid mean velocity (in future, Eulerian will always imply such a frame of reference), then the effect of a constant drift velocity $\mathbf{v}_g$ on particle diffusion is seen by replacing $U_{ij}^{(p)}(t)$ in (1.2) by $R_{ij}(\mathbf{v}_g t, 0)$. Yudine has referred to this phenomenon as the 'effect of crossing trajectories'. It is clear that such an effect will entirely dominate particle diffusion at arbitrarily large $\mathbf{v}_g$ when the time scale associated with $R_{ij}(\mathbf{v}_g t, 0)$ can be made arbitrarily small compared with the eddy decay time. For zero $\mathbf{v}_g$, $U_{ij}^{(p)}(t)$ reverts to the Lagrangian fluid-point correlation in Csanady's system. On this basis, Csanady describes the behaviour of $U_{11}^{(p)}(t)$ in the vertical direction due to eddy decay and crossing trajectories by the two numbers $v_0 t/l_1$ and $v_g t/l_1$ respectively, where v_0 is the intensity of the turbulence and l_1 the vertical integral length scale. By choosing a functional form for $U_{11}^{(p)}$ consistent with similar shapes for Eulerian spatial and Lagrangian fluid-point correlations, Csanady obtains a formula for $\epsilon_{11}^{(p)}(\infty)$ of the form

$$\epsilon_{11}^{(p)}(\infty) = \epsilon^{(f)}(\infty)\left\{\left(\frac{v_g}{v_0}\right)^2 + \frac{1}{r^2}\right\}^{-\frac{1}{2}}, \tag{1.3}$$

where the superscript f refers to the fluid and r is the ratio of the product of the Lagrangian integral time scale and v_0 to l_1. Using similar arguments Csanady also obtains a formula for the particle diffusion coefficient in the horizontal direction. Here, however, because of continuity of flow, the Eulerian length scale is different from the equivalent vertical scale. The relationship of scales existing in isotropic stationary turbulence is assumed and the form for $\epsilon_{22}^{(p)}(\infty)$ suggested by Csanady is

$$\epsilon_{22}^{(p)}(\infty) = \epsilon^{(f)}(\infty)\left\{4\left(\frac{v_g}{v_0}\right)^2 + \frac{1}{r^2}\right\}^{-\frac{1}{2}}. \tag{1.4}$$

Both formulae are presented here as a basis for future comparison. Clearly the effect of increasing v_g is to reduce the particle diffusion coefficient both in the direction parallel and in the direction normal to $\mathbf{v}_g$. In the limit $v_g/v_0 \to \infty$, the diffusion coefficient normal to $\mathbf{v}_g$ is a half that parallel to $\mathbf{v}_g$, both coefficients being inversely proportional to v_g, a result first obtained by Yudine (1959). It is worth pointing out that (1.1) was used as the basis of Csanady's analysis, and because of neglect of inertia effects $\langle \Delta v_i(0)\, \Delta v_j(t)\rangle$ was equivalent to $U_{ij}^{(p)}(t)$. However, because of the validity of (1.2) for linear drag, in the limit of large v_g the same results would apply for particles for which inertia effects were significant (Lumley 1976). However, in the absence of a constant drift these formulae are no longer valid except in the limit of zero inertia. It is reasonable to suppose that the effect of inertia in this instance is such that as the particle inertia is increased from zero $U_{ij}^{(p)}(t)$ changes smoothly from the fluid Lagrangian autocorrelation to the single-point Eulerian velocity–time correlation. Whether, on the basis of (1.2), the long-time particle diffusion coefficient is, in general, greater or less than that of the fluid clearly depends upon whether for the fluid the Eulerian integral time scale is greater or less than the Lagrangian integral time scale. The most notable theoretical work on this subject began with Corrsin's (1959) independence approximation, known as Corrsin's hypothesis, but latterly has consisted of the more convincing work of Kraichnan. Kraichnan (1964) has described Eulerian and Lagrangian velocity fields in which one would intuitively expect Lagrangian time scales to be less than the equivalent Eulerian time scales. This conclusion is corroborated by his computer simulations of fluid-point dispersion in a Gaussian random velocity field (Kraichnan 1970). It was also significant that good agreement was found between these 'exact' results and those obtained from the direct-interaction approximation. This Eulerian–Lagrangian time-scale relationship is also an implication of Phythian's (1975) formulation. Phythian's method uses a 'second approximation' to the solution of the equation of motion of a fluid point moving through a random isotropic homogeneous and stationary Gaussian velocity field. Considering the crudity of the approximation compared with the 'fully iterated' solution, it will appear surprising that, as far as the finally calculated fluid Lagrangian velocity autocorrelation is concerned, the results are in remarkably good agreement with Kraichnan's computer simulations for random velocity fields characteristic of real turbulence. As a basis for calculating random particle motion with finite inertia, it will be apparent in the next section that this approximation is strictly a perturbation about the motion of particles of large inertia. We should therefore expect the greatest discrepancy between real and approximated motion to occur in calculating fluid-point motion, for which the method was originally used. A comparison of Phythian's results and those from Kraichnan's numerical simulation over a range of Gaussian velocity fields indicates that the approximation is exact for velocity fields with δ-function correlation times, becoming progressively worse as one approaches both infinite Eulerian correlation times and δ-function energy spectra. Even in this extreme case, the difference is significant only in the negative region of the Lagrangian correlation function. That the agreement is not fortuitous is supported by the recent work of Lundgren & Pointin (1976), who have demonstrated a close relationship between Phythian's method and that of an approximation based on Corrsin's hypothesis. They have used this approximation to evaluate the same dispersion coefficients in the same random velocity fields as those chosen by Kraichnan and find exactly the same trend of agreement as that of Phythian. More significant here is that in all four Gaussian velocity fields considered their solution and those of Phythian differ by only a few per cent. Their value for the diffusion coefficient is obtained from a second-order differential equation in time for the fluid-point mean-square displacement, for which a solution by iteration is found to be rapidly convergent. Iterating the equation twice gives the function obtained from Phythian's approximation. This, in itself, is a remarkable result since it is not obvious that these approximations bear any formal resemblance to one another. Furthermore, Corrsin's hypothesis is clearly a less crude assumption than has previously been thought. Lundgren & Pointin suggest that in Gaussian fields such an assumption is acceptable in situations where the displacement of a fluid point is only weakly coupled to any one Fourier mode of the field. Certainly, this appears consistent with the error trend of the approximation when applied to the various random fields considered by Kraichnan. More recently, Weinstock (1976) has examined the strength of the approximation in arbitrary random velocity fields, deriving conditions of suitability involving third-order spatial correlations that seem well satisfied in homogenous turbulence.

It is the purpose of the analysis below to apply Phythian's technique to calculate $U_{ij}^{(p)}(t)$ and some important particle dispersion coefficients derivable from it for the case of particle dispersion in isotropic homogeneous and stationary turbulence. The analysis will incorporate both the simultaneous effects of finite particle inertia and crossing trajectories due to the action of a constant external gravitational force. The essential difference between this analysis and that of Csanady and others is that we shall consider particle dispersion in the absence of constant drift. Though the extrapolation to zero particle inertia may not appear formally justified within the spirit of the approximation, we believe that it is permissible for two reasons; first, we shall be applying it to a system where the approximation for fluid-point motion gives good agreement with results from a numerical simulation, and second, in a more general sense, the end results will be closely related to those obtained from an approximation based on Corrsin's hypothesis, which has been shown to be applicable in homogeneous turbulence. In the absence of a general transport equation it is assumed on the basis of the evidence for both passive-scalar and heavy-particle motion in such turbulence that Fick's law is operative throughout the entire range of particle inertia, and thus that the particle diffusion coefficient uniquely determines the temporal evolution of the particle concentration. The analysis is consistent with the relationship between Lagrangian and Eulerian correlation times formulated by Kraichnan and Phythian. In the absence of external forces it is to be expected, therefore, that in the long-time limit the particle diffusion coefficient will be greater than that of the fluid. This differs significantly from the earlier work of Tchen (1947) and Peskin (1962), where the long-time particle diffusion coefficient was either the same as (Tchen) or less than (Peskin) that of the fluid.

2. Analysis

Several assumptions are made on the grounds of mathematical simplicity. The drag force acting upon a particle as it moves through the fluid is assumed to be linear in its velocity relative to that of the fluid at the same point and time. The particles are considered sufficiently large that Brownian diffusion can be neglected in comparison with the transport originating in the interaction of the particle with the turbulent velocity field. The fluid is endowed with a constant mean flow, the turbulent fluctuations superimposed upon this flow being considered isotropic, stationary and homogeneous throughout a frame of reference moving with the mean velocity.

The assumption of linear drag is strictly applicable only to uniform particle motion for which the particle Reynolds number is less than unity. (For a more general equation of motion in turbulence and the restrictions which apply see Hinze 1959.) We have thus implicitly assumed that, in the statistical correlations we wish to calculate, the linear drag is dominant over those drag forces arising from the particles' acceleration, e.g. the Basset history force. This is not such an implausible assumption. Although such 'acceleration' forces are significant in a particle's transient response to fluid flow, it has been shown that in a statistically stationary response (such as is desired when calculating turbulent dispersion) the effect of the Basset force can be neglected (Ahmadi & Goldschmidt 1971). In any case for particle Reynolds numbers less than unity the analysis is capable of including such forces and we shall discuss this at a later stage. For unrestricted motion the functional dependence of the drag force on particle and fluid variables is extremely complicated though under appropriate restrictions it can be linearized and approximated by the quasi-steady form assumed here (Lumley 1957). In parentheses, we may plausibly argue that, for the statistical quantities we wish to calculate, the dominant contribution is from the particle's response to the large scales of fluid motion, for which the linear drag is asymptotically correct.

We thus consider the motion of a particle throughout such a turbulent field in a frame of reference moving with the mean velocity of the flow. The fluid velocity is specified by a Eulerian velocity field $\mathbf{u}(\mathbf{x},t)$, where $(\mathbf{x},t)$ are the space–time co-ordinates of the field. In general, the particle may also be acted upon by some external force field, which we shall characterize by a particle acceleration $\mathbf{f}_e(\mathbf{x},t)$. Thus if $\mathbf{v}(t)$ is the velocity of a particle at time t, then the equation of motion of the particle is

$$\dot{\mathbf{v}}(t)+\beta\mathbf{v}(t) = \beta\mathbf{u}(\mathbf{X}(t),t)+\mathbf{f}_e(\mathbf{X}(t),t), \tag{2.1}$$

where β is a constant for the motion (β^{-1} is referred to hereafter as the particle relaxation time and quantifies its inertia) and $\mathbf{X}(t)$ is the position of the particle in the field at time t, which is explicitly given by

$$\mathbf{X}(t) = \int_0^t \mathbf{v}(\tau)\,d\tau, \tag{2.2}$$

where without loss of generality the position of the particle at time zero is taken to be the origin of the field co-ordinates. In this analysis we assume that the only external force acting on the particle is gravity, so that $\mathbf{f}_e$ is constant and identical to $\mathbf{g}$, the acceleration due to gravity. Equations (2.1) and (2.2) constitute a set of nonlinear equations in v_i for which there is in general no analytic solution. The stochastic nature of the problem is reflected in the vector $\mathbf{u}(\mathbf{x},t)$. Knowledge of the particle dispersion in this field would require a general solution to this set of equations for any realization of the fluid velocity field $\mathbf{u}(\mathbf{x},t)$ together with its statistical description. It seems natural to represent the statistical nature of this field by either a probability density functional $P[\mathbf{u}(\mathbf{x},t)]$ or its associated characteristic functional $\langle \exp[i\int \boldsymbol{\phi}(x,t).\mathbf{u}(\mathbf{x},t)\,d\mathbf{x}\,dt]\rangle$. Complete knowledge of either of these quantities implies a solution to the closure problem of turbulence, for which no entirely satisfactory solution exists as yet. In the absence of a method for deriving the complete nature of $P[\mathbf{u}(\mathbf{x},t)]$ from the moment equations of turbulence, it is assumed, as in Phythian's formulation, that the probability distribution for $\mathbf{u}(\mathbf{x},t)$ is Gaussian. Although lacking certain important features of real turbulent flow, namely nonlinear energy transfer and the passive convection of small spatial scales of motion by larger ones, this distribution appears to represent quite well the statistics of large scales of motion, to which the fluid dispersions referred to below are more sensitive (Frenkiel & Klebanoff 1967*a*, *b*) and indeed to which particle dispersion, in general, will be doubly more sensitive because of the built-in selective response to low frequency fluctuations.

In the situation considered here, a particle is allowed to come to equilibrium with the fluid so that the particle motion has lost all memory of its initial conditions. All subsequent evolution of the correlated motion of the particle will be independent of whatever time axis is chosen to describe the process. The state of motion of the particle in this instance can be conveniently represented by

$$\mathbf{v}(t) = \beta\int_{-\infty}^{t} \mathbf{u}(\mathbf{X}(\tau),\tau)\,e^{\beta(\tau-t)}d\tau+\mathbf{v}_g, \tag{2.3}$$

where $\mathbf{v}_g = \mathbf{g}/\beta$ is commonly referred to as the settling velocity. A solution to this equation may be obtained by generating a chain of approximants for $\mathbf{u}(\mathbf{X}(\tau),\tau)$ based on an initial approximation $\mathbf{X}^{(0)}(\tau)$ for $\mathbf{X}(\tau)$. The first approximant for $\mathbf{u}(\mathbf{X}(t),\tau)$ is simply

$$\mathbf{u}^{(1)}(\tau) = \mathbf{u}(\mathbf{X}^{(0)}(\tau),\tau), \tag{2.4}$$

so that the first approximant for $\mathbf{X}(t)$, obtained from (2.3) with $\mathbf{u}^{(1)}(\tau)$ for $\mathbf{u}(\mathbf{X}(\tau),\tau)$, is thus

$$\mathbf{X}^{(1)}(t) = \mathbf{v}_g t + \int \mathbf{u}(\mathbf{X}^{(0)}(\tau),\tau)\{\Theta(\tau-t)\,\psi(\tau-t) - \Theta(\tau)\,\psi(\tau)\}\,d\tau, \tag{2.5}$$

where

$$\psi(\tau) = 1 - e^{\beta\tau}, \quad \Theta(\tau) = \begin{cases} 1, & \tau \leqslant 0, \\ 0, & \tau > 0. \end{cases} \tag{2.6}$$

In this equation, as in future equations where the range of integration is not indicated explicitly, it is assumed to be over the entire space of the variable.

The second approximant for $\mathbf{u}(\mathbf{X}(\tau),\tau)$ is generated by setting $\mathbf{X}(\tau) = \mathbf{X}^{(1)}(\tau)$; explicitly

$$\mathbf{u}^{(2)}(\tau) = \mathbf{u}(\mathbf{v}_g\tau + \int d\tau'\mathbf{u}(\mathbf{X}^{(0)}(\tau'),\tau')\{\Theta(\tau'-\tau)\,\psi(\tau'-\tau) - \Theta(\tau')\,\psi(\tau')\},\tau) \tag{2.7}$$

and hence

$$\mathbf{X}^{(2)}(t) = \mathbf{v}_g t + \int \mathbf{u}(v_g\tau + \int d\tau'\mathbf{u}(\mathbf{X}^{(0)}(\tau'),\tau')\{\Theta(\tau'-\tau)\,\psi(\tau'-\tau) - \Theta(\tau')\,\psi(\tau')\},\tau) \times \{\Theta(\tau-t')\,\psi(\tau-t) - \Theta(\tau)\,\psi(\tau)\}\,d\tau \tag{2.8}$$

and so on, so that, in general, the nth approximants for $\mathbf{u}^{(n)}(\tau)$ and $\mathbf{X}^n(t)$ are simply

$$\mathbf{u}^{(n)}(\tau) = \mathbf{u}(\mathbf{X}^{(n-1)}(\tau),\tau), \tag{2.9}$$

$$\mathbf{X}^{(n)}(t) = \mathbf{v}_g t + \int \mathbf{u}^{(n)}(\tau)\{\Theta(\tau-t)\,\psi(\tau-t) - \Theta(\tau)\,\psi(\tau)\}\,d\tau. \tag{2.10}$$

The state of motion of the particle implies that the particle kinematic variables are stationary random variables. Comparison of the left-hand with the right-hand side in (2.1) with $\mathbf{f}_e = \mathbf{g}$ implies that $\mathbf{u}(\mathbf{X}(t),t)$ is also stationary and random. Returning to (2.3), it is obvious therefore that

$$\langle \mathbf{v}(t)\rangle = \mathbf{v}_g. \tag{2.11}$$

Consider thus the correlation $\langle \Delta v_i(t_1)\,\Delta v_j(t_2)\rangle$ of particle velocity fluctuations $\Delta\mathbf{v}$ relative to $\mathbf{v}_g$, i.e. $\Delta\mathbf{v}(t) = \mathbf{v}(t) - \mathbf{v}_g$. From (2.3),

$$\langle \Delta v_i(t_1)\,\Delta v_j(t_2)\rangle = \beta^2 \exp[-\beta(t_1+t_2)] \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \exp[\beta(\tau_1+\tau_2)] \times \langle u_i(\mathbf{X}(\tau_1),\tau_1)\,u_j(\mathbf{X}(\tau_2),\tau_2)\rangle\,d\tau_1\,d\tau_2. \tag{2.12}$$

Because of stationarity $\langle u_i(\mathbf{X}(\tau_1),\tau_1)\,u_j(\mathbf{X}(\tau_2),\tau_2)\rangle$ is a function of only $\tau_1-\tau_2$ in time and there will be no loss of generality if we replace it by

$$\langle u_i(0,0)\,u_j(\mathbf{X}(\tau_1-\tau_2),\tau_1-\tau_2)\rangle$$

in (2.11). This procedure may seem trivial at this stage, but when we eventually replace $\langle u_i(0,0)\,u_j(\mathbf{X}(\tau_1-\tau_2),\tau_1-\tau_2)\rangle$ by an approximation, it ensures that the expression for $\langle \Delta v_i(t_1)\,\Delta v_j(t_2)\rangle$ is a function only of t_1-t_2 in time, consistent with the real $\langle \Delta v_i(t_1)\,\Delta v_j(t_2)\rangle$. It is easily shown that

$$\langle \Delta v_i(t_1)\,\Delta v_j(t_2)\rangle = \frac{\beta}{2}\int d\xi\{\theta(\xi+t_1-t_2)\exp[\beta(\xi+t_1-t_2)] + \theta(\xi-t_1+t_2) \times \exp[\beta(\xi-t_1+t_2)]\}\,\langle u_i(0,0)\,u_j(\mathbf{X}(\xi),\xi)\rangle, \tag{2.13}$$

which is consistent with the initial assumption that the correlated motion of the particle is independent of the time axis. $\langle u_i(0,0)\, u_j(\mathbf{X}(\xi),\xi)\rangle$ is readily identified with the quantity $U_{ij}^{(p)}(\xi)$, defined in (1.2). On the basis of Phythian's calculations, we consider $\langle u_i(0,0)\, u_j^{(2)}(t)\rangle$ to be a sufficiently good approximation to $\langle u_i(0,0)\, u_j(\mathbf{X}(t),t)\rangle$. In this instance our initial approximation for $\mathbf{X}(t)$ is $\mathbf{X}^{(0)}(t) = \mathbf{v}_g t$, so that explicitly

$$\langle u_i(0,0)\, u_j^{(2)}(t)\rangle = \int d\mathbf{x}\, \langle u_i(0,0)\, u_j(\mathbf{x},t)\, \delta(\mathbf{x}-\mathbf{v}_g t - \int d\tau\, \mathbf{u}(\mathbf{v}_g \tau,\tau)\, q(t,\tau))\rangle \tag{2.14}$$

$$= \frac{1}{(2\pi)^3}\int d\mathbf{x}\int d\mathbf{k}\exp(-i\mathbf{k}.\mathbf{v}_g t)\Big\langle u_i(0,0)\, u_j(\mathbf{x},t) \times \exp\left(-i\mathbf{k}.\int d\tau\, \mathbf{u}(\mathbf{v}_g\tau,\tau)\, q(t,\tau)\right)\Big\rangle, \tag{2.15}$$

where, for convenience, we have used $q(t,\tau)$ for $\Theta(\tau-t)\,\psi(\tau-t)-\Theta(\tau)\,\psi(\tau)$. We note here that the approximation $\langle u_i(0,0)\, u_j^{(1)}(t)\rangle$ forms the basis of the approximation used by Csanady, Yudine and Meek, in which the particle propagator in (2.14) is simply $\delta(\mathbf{x}-\mathbf{v}_g t)$ and thus is restricted to heavy particles for which v_g greatly exceeds the intensity of the turbulence.

On the basis that the distribution of $\mathbf{u}(\mathbf{x},t)$ is Gaussian with zero mean, we write the characteristic functional

$$M[\boldsymbol{\phi}(\mathbf{y},\tau)] \equiv \langle \exp[i\int d\mathbf{y}\int d\tau\, \boldsymbol{\phi}(\mathbf{y},\tau).\mathbf{u}(\mathbf{y},\tau)]\rangle$$

explicitly as

$$M[\boldsymbol{\phi}(\mathbf{y},\tau)] = \exp[-\tfrac{1}{2}\int\int \phi_i(\mathbf{y}_1,\tau_1)\, R_{ij}(\mathbf{y}_1-\mathbf{y}_2,\tau_1-\tau_2)\, \phi_j(\mathbf{y}_2,\tau_2)\, d\mathbf{y}_1 d\tau_1 d\mathbf{y}_2 d\tau_2], \tag{2.16}$$

where R_{ij} is the correlation function of the fluid velocity field, defined by

$$\langle u_i(\mathbf{x},t)\, u_j(\mathbf{x}',t')\rangle = R_{ij}(\mathbf{x}-\mathbf{x}',t-t'). \tag{2.17}$$

If we assume that R_{ij} is separable in $\mathbf{x}$ and t, then in isotropic, homogeneous and stationary turbulence R_{ij} is constrained to the form

$$R_{ij}(\mathbf{x},t) = \frac{1}{4\pi}\int d\mathbf{k}\left(\delta_{ij}-\frac{k_i k_j}{k^2}\right)\frac{E(k)}{k^2}\, e^{-i\mathbf{k}.\mathbf{x}} D(t), \tag{2.18}$$

with

$$D(0) = 1, \quad \int_0^\infty dk E(k) = \tfrac{3}{2}v_0^2.$$

The ensemble average contained within the integrand of (2.14) is formally

$$-\frac{\delta^2}{\delta\phi_i\,\delta\phi_j} M[\boldsymbol{\phi}; 0, 0, \mathbf{x}, t],$$

with

$$\boldsymbol{\phi}(\mathbf{y},\tau) = -\mathbf{k}q(t,\tau)\,\delta(\mathbf{y}-\mathbf{v}_g\tau). \tag{2.19}$$

Explicitly evaluating this expectation value using the expression given for a Gaussian characteristic functional and employing the incompressibility condition yields the following expression for $\langle u_i(0,0)\, u_j^{(2)}(t)\rangle$:

$$\langle u_i(0,0)\, u_j^{(2)}(t)\rangle = \int d\mathbf{k}\, \tilde{Q}_{ij}(\mathbf{k},t)\exp(-\tfrac{1}{2}k_i k_j \int\int d\tau_1 d\tau_2\, q(t,\tau_1)\, Q_{ij}(0,\tau_1-\tau_2)\, q(t,\tau_2)), \tag{2.20}$$

where

$$Q_{ij}(\mathbf{x},t) = R_{ij}(\mathbf{v}_g t+\mathbf{x},t).$$

Employing the expression for $q(t, \tau)$ and reducing gives finally

$$\langle u_i(0,0)\, u_j^2(t)\rangle = \int d\mathbf{k}\, \tilde{Q}_{ij}(\mathbf{k}, t) \exp\left(-k_i k_j \alpha_{ij}(t, \beta)\right), \tag{2.21}$$

where

$$\alpha_{ij}(t,\beta) = \int_0^t \left[t - \xi + \frac{1}{\beta}\sinh\beta(\xi - t)\right] Q_{ij}(0,\xi)\, d\xi + \frac{1}{\beta}\left[\cosh\beta t - 1\right] \int_0^\infty e^{-\beta\xi} Q_{ij}(0,\xi)\, d\xi. \tag{2.22}$$

The inertial dependence of $\langle u_i(0,0)\, u_j^{(2)}(t)\rangle$ is simply reflected in the behaviour of $\alpha_{ij}(t,\beta)$. As $\beta \to 0$ (high inertia), $\alpha_{ij}(t,\beta) \to 0$, i.e. the particle is forced into a Eulerian framework moving with a velocity $\mathbf{v}_g$ relative to the zero-mean-velocity frame of reference. At the other extreme, $\beta \to \infty$ (particle motion equivalent to fluid motion), $\mathbf{v}_g = 0$ and

$$\alpha_{ij}(t,\beta) \to v_0^2 \int_0^t \{t - \xi\}\, D(\xi)\, d\xi,$$

giving a form for $\langle u_i(0,0)\, u_j^{(2)}(t)\rangle$ which is identical to Phythian's expression for a fluid particle. Note also that only in the extreme case when particle and fluid-point motion are equivalent is the dispersion isotropic. Furthermore, if we were to ignore the effect of gravity, the time scale associated with $\langle u_i(0,\,0)\, u_j^{(2)}(t)\rangle$ for any β would be less than that for $\langle u_i(0,\,0)\, u_j^{(2)}(t)\rangle$ when $\beta = 0$. Thus the qualitative features of $\langle u_i(0,0)\, u_j^{(2)}(t)\rangle$ appear to be consistent with the remarks made in the previous section.

3. Evaluation of dispersion coefficients for assumed $E(k)$ and $D(t)$

From the results of the previous section some important statistical quantities associated with the collective particle motion are evaluated using a Gaussian for $D(t)$, namely

$$D(t) = \exp\left(-\tfrac{1}{2}\omega_0^2 t^2\right), \tag{3.1}$$

and a form for $E(k)$ identical with that used by Phythian for which he obtains excellent agreement with Kraichnan's numerical results, namely

$$E(k) = 16(2/\pi)^{\frac{1}{2}} v_0^2 k_0^{-5} k^4 \exp\left(-2k^2/k_0^2\right). \tag{3.2}$$

The form of $E(k)$ is consistent with k being distributed isotropically in $\mathbf{k}$ space on the surface of a sphere of radius k_0, the occurrence of each component of $\mathbf{k}$ being determined by a Gaussian distribution of standard deviation $\frac{1}{2}k_0$ (Kraichnan 1970). It is not expected that a universal or even a simple form exists for the energy spectrum of real isotropic turbulence. The form for $E(k)$ used here is asymptotically correct at low wavenumbers but is in serious error at high wavenumbers. However, the quantities considered here tend to be dominated by the large scales of motion, especially at large times, so that the precise form of $E(k)$ at high wavenumbers ought to be relatively unimportant. The choice of $D(t)$ is somewhat arbitrary: on dimensionless grounds ω_0 is a function of $k_0 v_0$. For the purpose of some of his calculations, Kraichnan arbitrarily sets $\omega_0 = k_0 v_0$; in this analysis we assume a constant of proportion γ, i.e.

$$\omega_0 = \gamma k_0 v_0. \tag{3.3}$$

We do this because it ought to be possible to ascribe some kind of value to γ from experimental measurements.

For convenience of presentation, the following dimensionless parameters are used:

$$\tau = k_0 v_0 t, \quad \lambda_g = v_g/v_0, \quad \beta = \beta'/k_0 v_0, \tag{3.4}$$

where β' is identical to the β defined in (2.1), and

$$\mu_{ij} = 1 + \tfrac{1}{2}k_0^2\,\alpha_{ij}(t,\beta). \tag{3.5}$$

It is natural to let $\mathbf{v}_g$ define the $i = 1$ axis, so that $[i = 1,2,3]$ forms a set of principal axes. In this instance

$$\left.\begin{aligned} Q_{11}(0,\tau) &= v_0^2\exp\left(-\tfrac{1}{2}\sigma^2\tau^2\right),\\ Q_{22}(0,\tau) &= Q_{33}(0,\tau) = v_0^2(1-\tfrac{1}{8}\lambda_g^2\tau^2)\exp\left(-\tfrac{1}{2}\sigma^2\tau^2\right),\\ Q_{ij}(0,\tau) &= 0, \quad i \neq j, \end{aligned}\right\} \tag{3.6}$$

where

$$\sigma^2 = \gamma^2 + \tfrac{1}{4}\lambda_g^2. \tag{3.7}$$

Using the adopted forms for $E(k)$ and $D(t)$ and with reference to the function

$$f(\tau) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\exp\left(\beta^2/2\sigma^2\right)}{2\sigma\beta}\, e^{\beta\tau}\operatorname{erfc}\frac{\sigma}{2^{\frac{1}{2}}}\left(\tau + \frac{\beta}{\sigma^2}\right),$$

the μ_{ij} are explicitly given by

$$\left.\begin{aligned} \mu_{11} &= 1 + \frac{1}{2}\left\{h(\tau) + \frac{1}{\sigma^2}\left[\exp\left(-\sigma^2\tau^2\right) - 1\right] + \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\tau\sigma\operatorname{erf}\frac{\sigma\tau}{2^{\frac{1}{2}}}\right\},\\ \mu_{33} &= \mu_{22} = 1 + \left(1 - \frac{\lambda_g^2}{8\sigma^2}\right)(\mu_{11} - 1) - \frac{\beta^2}{16\sigma^4}h(\tau), \end{aligned}\right\} \tag{3.8}$$

where

$$h(\tau) = f(\tau) + f(-\tau) - 2f(0). \tag{3.9}$$

For convenience, we redefine $U_{ij}^{(p)}(\tau) \equiv \langle u_i(0,0)\, u_j^{(2)}(\tau)\rangle$, so that in this instance

$$\left.\begin{aligned} U_{11}^{(p)}(\tau) &= \frac{v_0^2\,D(\tau)}{\mu_2^{22}\mu_{11}^{\frac{1}{2}}}\exp\left(-\frac{1}{8}\frac{\lambda_g^2\tau^2}{\mu_{11}}\right),\\ U_{33}^{(p)}(\tau) &= U_{22}^{(p)}(\tau) = \frac{1}{2}\left\{1 + \frac{\mu_{22}}{\mu_{11}}\left(1 - \frac{\lambda_g^2\tau^2}{4\mu_{11}}\right)\right\}U_{11}^{(p)}(\tau),\\ U_{ij}^{(p)}(\tau) &= 0, \quad i \neq j. \end{aligned}\right\} \tag{3.10}$$

The particle velocity correlation function and diffusion coefficient are both derivable from $U_{ij}^{(p)}(\tau)$. The particle velocity correlation function $\langle \Delta v_i(0)\,\Delta v_j(\tau)\rangle$ is simply

$$\langle \Delta v_i(0)\,\Delta v_j(\tau)\rangle = \frac{\beta}{2}\int_0^\infty d\xi\, e^{-\beta\xi}\{U_{ij}^{(p)}(\xi+\tau) + U_{ij}^{(p)}(\xi-\tau)\}, \tag{3.11}$$

from which the mean-square velocity of the particle

$$\langle \Delta v_i\,\Delta v_j\rangle = \beta\int_0^\infty d\xi\, e^{-\beta\xi}\, U_{ij}^{(p)}(\xi). \tag{3.12}$$

The diffusion coefficient $\epsilon_{ij}^{(p)}(\tau)$ is similarly

$$\begin{aligned} \epsilon_{ij}^{(p)}(\tau) &= \frac{\beta}{2k_0 v_0}\int_0^\infty d\xi\, e^{-\beta\xi}\int_{\xi-\tau}^{\xi+\tau} d\eta\, U_{ij}^{(p)}(\eta)\\ &= \frac{1}{k_0 v_0}\sinh\beta\tau\int_0^\infty d\eta\, e^{-\beta\eta}\,U_{ij}^{(p)}(\eta) + \frac{1}{k_0 v_0}\int_0^\tau d\eta\,(1 - e^{-\beta\tau}\cosh\beta\eta)\,U_{ij}^{(p)}(\eta), \end{aligned} \tag{3.13}$$

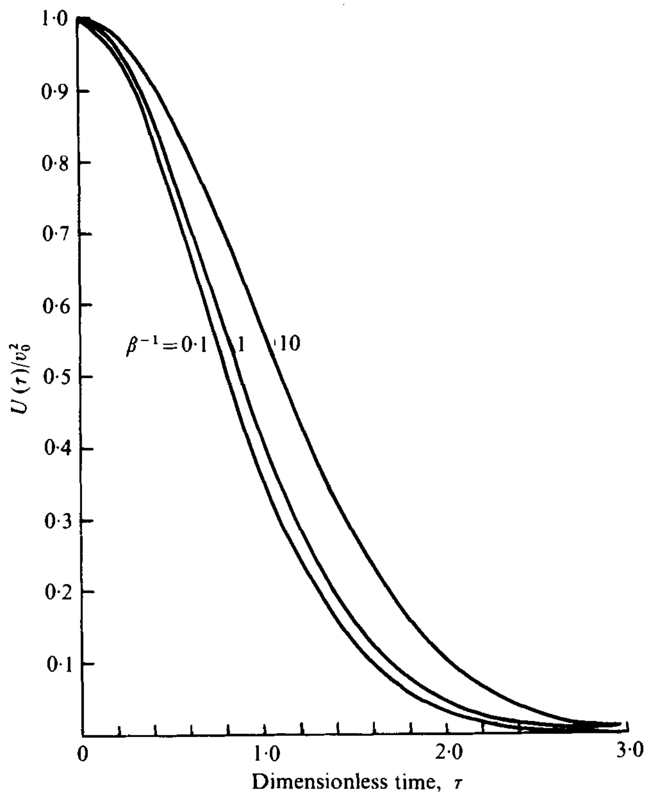

FIGURE 1. Fluid velocity autocorrelation $U^{(p)}(\tau)$ along a particle trajectory; $\gamma = 1$, $1/F_g = 0$.

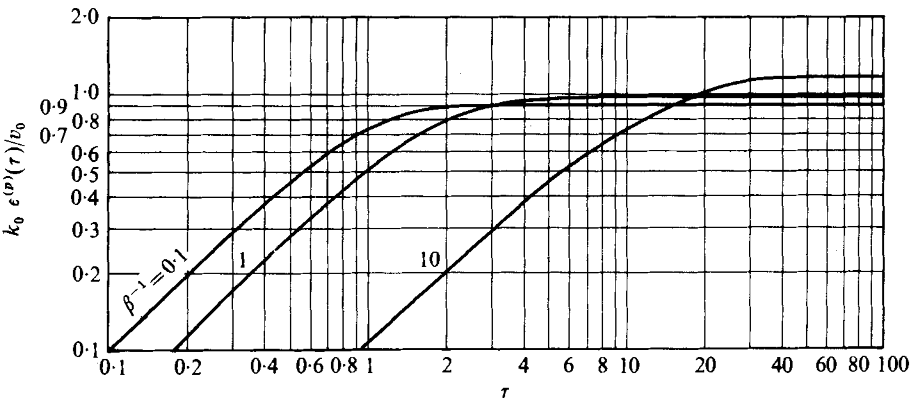

FIGURE 2. Particle diffusion coefficient $\epsilon^{(p)}(\tau)$; $\gamma = 1$, $1/F_g = 0$.

which when $r \to \infty$ degenerates to

$$\epsilon_{ij}^{(p)}(\infty) = \frac{1}{k_0 v_0}\int_0^\infty U_{ij}^{(p)}(\eta)\,d\eta. \tag{3.14}$$

The particle integral time scale $\tau_{ij}^{(p)}$ associated with $\langle \Delta v_i(0)\,\Delta v_j(\tau)\rangle$ is clearly given by

$$\tau_{ij}^{(p)} = (k_0 v_0 \langle \Delta v_i \Delta_j \rangle)^{-1}\int_0^\infty U_{ij}^{(p)}(\eta)\,d\eta. \tag{3.15}$$

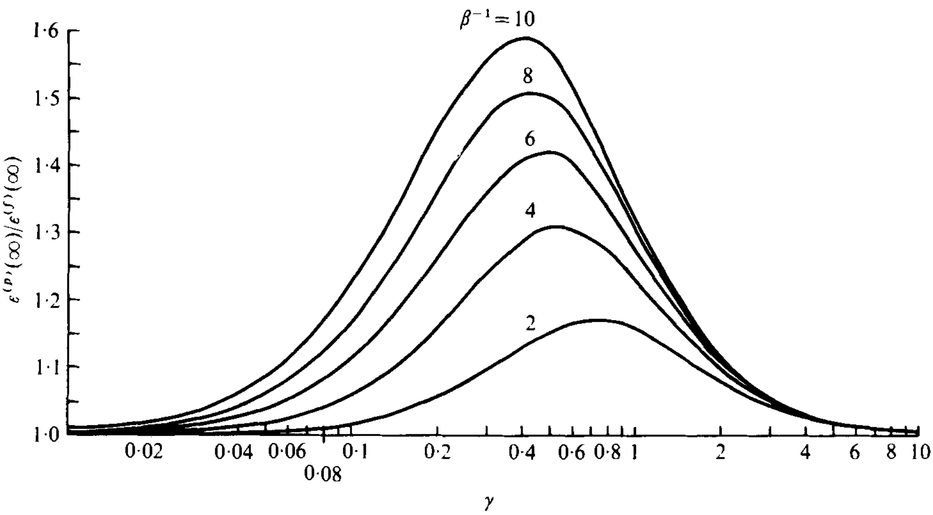

FIGURE 3. Ratio of long-time particle and fluid diffusion coefficients $\epsilon^{(p)}(\infty)/\epsilon^{(f)}(\infty)$ as a function of γ, the ratio of the eddy circulation time to the Eulerian velocity-field correlation time; $1/F_g = 0$.

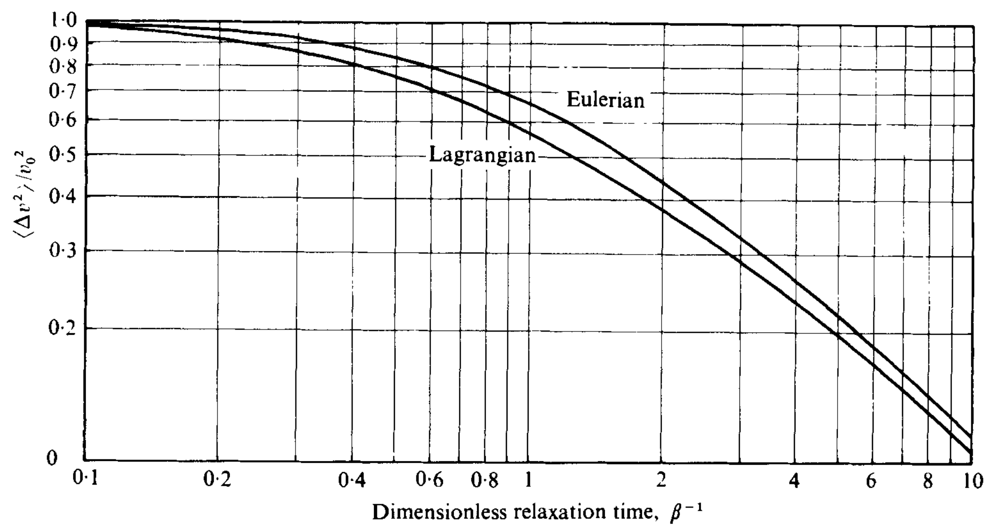

FIGURE 4. Particle mean-square velocity relative to mean based on Eulerian and Lagrangian time scales; $\gamma = 1$, $1/F_g = 0$.

To account for the dependence of λ_g on β, we define a number $F_g = k_0 v_0^2/g$, so that

$$\lambda_g = [F_g \beta]^{-1}. \tag{3.16}$$

F_g is akin to the Froude number, but is defined here in terms of a length scale and intensity of the turbulence, rather than the mean velocity and equivalent diameter of the flow. For illustrative purposes and for direct comparison with Phythian's calculations on fluid-point dispersion, the quantities referred to above have been evaluated for the arbitrary case of $\gamma = 1$, unless otherwise stated. The simplest and most interesting case, considered first, is particle dispersion in an isotropic turbulent fluid where the

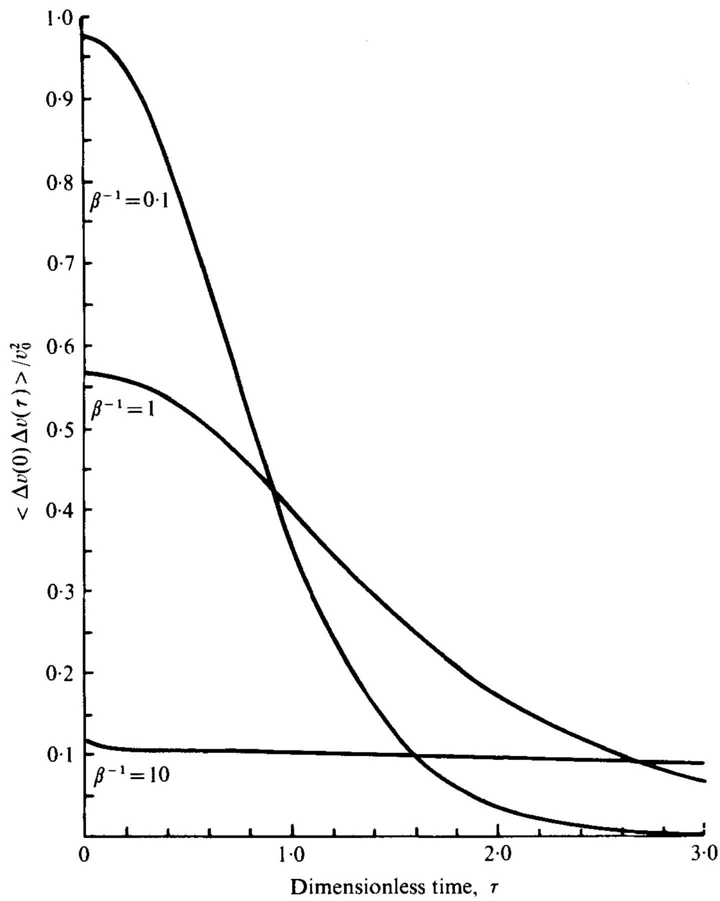

FIGURE 5. Particle velocity autocorrelation $\langle \Delta v(0)\ \Delta v(\tau)\rangle$; $\gamma = 1$, $1/F_g = 0$.

effect of gravity can be ignored ($1/F_g \sim 0$). In this instance, the dispersion is totally isotropic, the indices i and j becoming superfluous to the process. The effect of increasing particle relaxation time on $U^{(p)}(\tau)$ (figure 1) is to force the particle more and more into an Eulerian frame of reference, so that as time increases the particle diffusion coefficient $\epsilon^{(p)}(\tau)$ eventually reaches a value which is consistently higher than the equivalent long-time fluid-point diffusion coefficient $\epsilon^{(f)}(\infty)$ (figure 2). For any given β^{-1}, the ratio $\epsilon^{(p)}(\infty)/\epsilon^{(f)}(\infty)$ exhibits a maximum as a function of γ (figure 3), approaching unity as $\gamma \to 0$ (frozen Eulerian field) and as $\gamma \to \infty$. In the latter limit the Eulerian and Lagrangian fields are coincident, the process is Markovian and the difference is zero (equivalent to the motion of molecules in a gas). In this instance the approximation we have used is asymptotically exact. Figure 4 shows curves of particle mean-square velocity as a function of particle relaxation time obtained by using a Lagrangian description of $U^{(p)}(\tau)$ [equations (4.10)] and also by treating $U^{(p)}(\tau)$ as the single-point Eulerian velocity correlation $v_0^2 D(\tau)$. In the limits of β^{-1} approaching zero and infinity the curves are coincident. The curves of the particle correlation function for various particle relaxation times (figure 5) are consistent with the notion of the persistence of particle velocity with increasing relaxation time. The remaining figures illustrate the

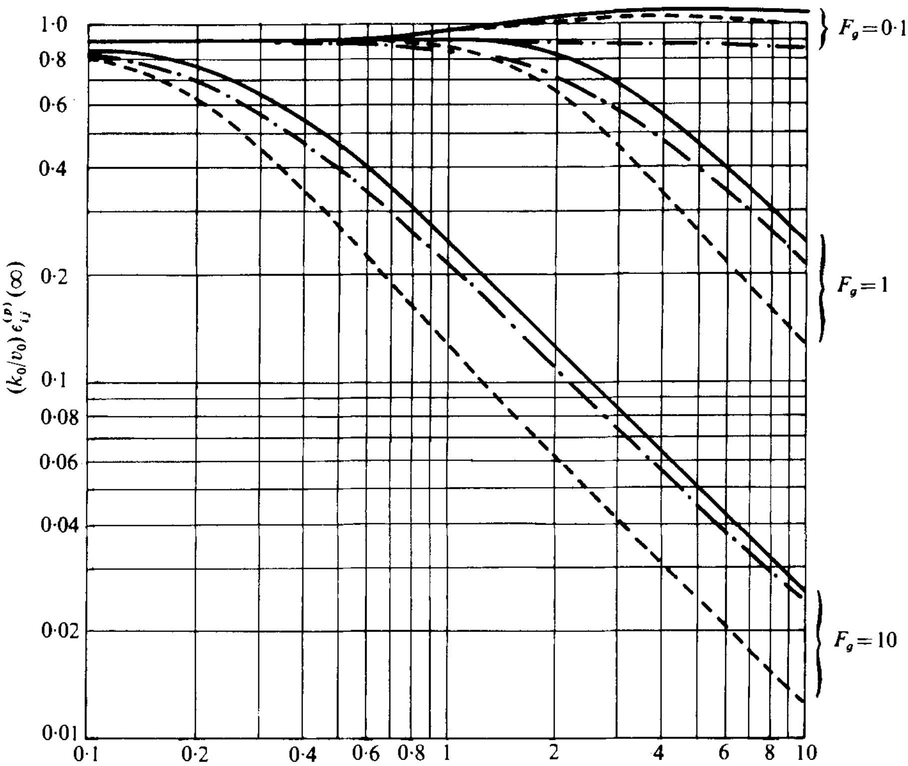

FIGURE 6. Long-time particle diffusion coefficient $\epsilon_{ij}^{(p)}(\infty)$; $\gamma = 1$. ——, $\epsilon_{11}^{(p)}(\infty)$, this theory; ---, $\epsilon_{22}^{(p)}(\infty)$, this theory; —·—, $\epsilon_{11}^{(p)}(\infty)$, equation (3.18), Csanady.

effect of crossing trajectories on the system. Here, with increasing particle relaxation time, $U_{ij}^{(p)}(\tau)$ approaches the Eulerian covariance $Q_{ij}(0,\tau)$ consistent with the particle travelling with a velocity $\mathbf{v}_g$ relative to the zero-mean-velocity frame of reference. In this instance, $U_{22}^{(p)}(\tau)$ is consistently less than $U_{11}^{(p)}(\tau)$, such that in the limit $\lambda_g \to \infty$ the integral time scales associated with $U_{22}^{(p)}(\tau)$ and $U_{11}^{(p)}(\tau)$ are in the ratio 1 : 2, the ratio of the lateral and longitudinal macroscales of the turbulence. It is clear that in this limit

$$\epsilon_{22}^{(p)}(\infty)/\epsilon_{11}^{(p)}(\infty) = \langle \Delta v_2^2 \rangle / \langle \Delta v_1^2 \rangle = 0{\cdot}5, \qquad (3.17)$$

a result in agreement with that of Yudine and Csanady; see (1.3) and (1.4). The increase in $\epsilon_{ij}^{(p)}(\infty)$ with increasing relaxation time is observable only for large F_g, over a restricted range of β^{-1}, and the coefficient finally declines as λ_g becomes dominant (figure 6). For comparison, we have also shown the values of $\epsilon_{11}(\infty)$ obtained from (1.3) (Csanady), using the value of r relevant to this system: explicitly,

$$\frac{k_0}{v_0}\,\epsilon_{11}^{(p)}(\infty) = \frac{0{\cdot}9}{[1+(0.359\lambda_g)^2]^{\frac{1}{2}}}. \qquad (3.18)$$

The values of $\epsilon_{11}^{(p)}(\infty)$ obtained from this formula are always consistently less than the equivalent values obtained from our theory, reflecting the fact that the diffusion coefficient is greater than that for the fluid for $\lambda_g = 0$. Our theory and that of Csanady

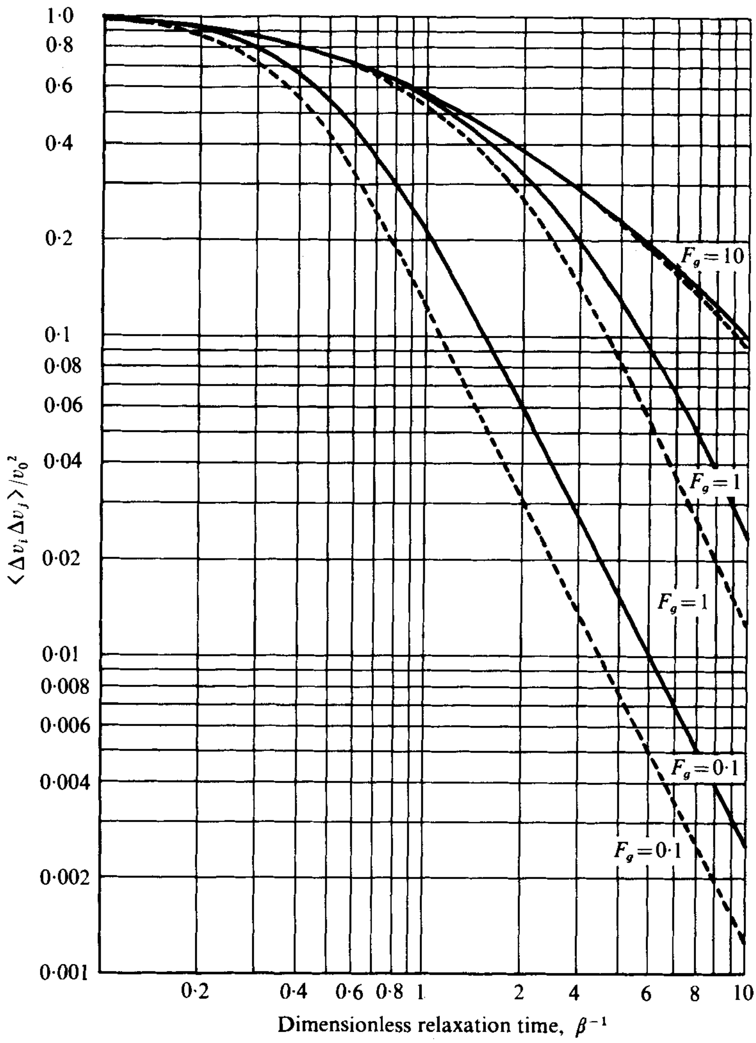

FIGURE 7. Particle mean-square velocity $\langle \Delta v_i \, \Delta v_j \rangle$; $\gamma = 1$. ——, $\langle \Delta v_1 \rangle^2$; ---, $\langle \Delta v_2 \rangle^2$.

give identical values for $\epsilon_{ij}^{(p)}(\infty)$ at infinite λ_g. Similarly, the value of F_g has a marked effect on the dependence of $\langle \Delta v_i^2 \rangle$ on β^{-1} (figure 7). Most interesting of all is the effect of F_g on the particle time scale $\tau_{ij}^{(p)}$ (figure 8). In situations in which the Eulerian frame of reference remains fixed and independent of β^{-1}, we should expect $\tau_{ij}^{(p)}$ to be monotonically increasing with increasing particle relaxation time. But in situations where F_g is not infinite, $\tau_{ij}^{(p)}$ initially declines from its value when $\beta^{-1} = 0$ to a minimum value dependent upon the strength of F_g. Within this range the decline of $\tau_{ij}^{(p)}$ is indicative of the fractional rate of decline of $\langle \Delta v_i \, \Delta v_j \rangle$ being less than that of the time scale associated with $U_{ij}^{(p)}(\tau)$ [equation (3.15)]. The effect is more pronounced in the direction normal to $\mathbf{v}_g$, increasing as F_g is reduced. It is significant that this effect has been observed in measurements of the particle velocity correlation $\langle \Delta v_2(0) \, \Delta v_2(t) \rangle$ in grid-generated turbulence (Snyder & Lumley 1971). Although the turbulence generated in this experiment was homogeneous and very nearly isotropic, it was not, however, station-

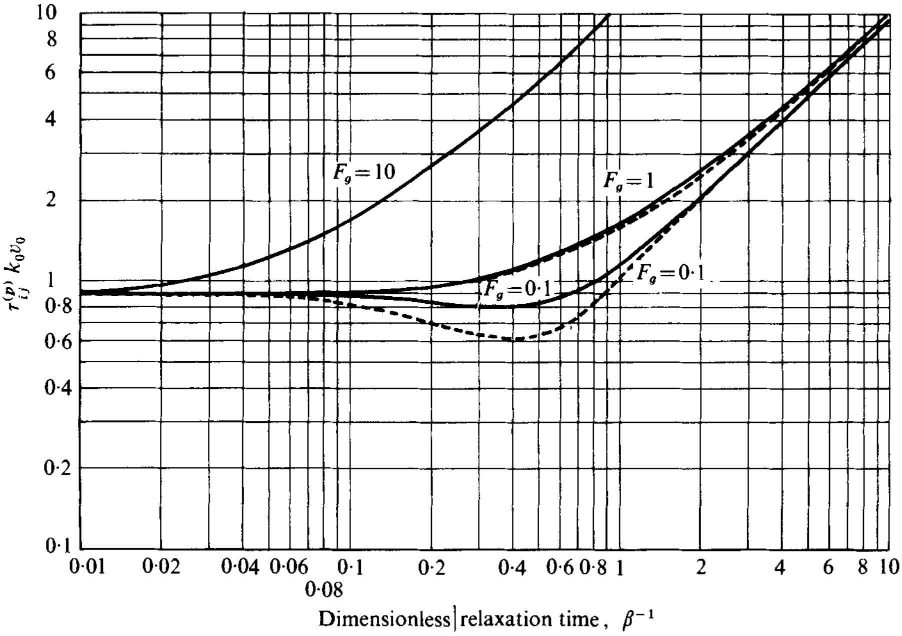

FIGURE 8. Particle integral time scale $\tau_{ij}^{(p)}$; $\gamma = 1$. ——, $\tau_{11}^{(p)}$; ---, $\tau_{22}^{(p)}$.

ary in real time and it is therefore not stricty valid to compare in detail this theory with Snyder & Lumley's experimental data. It is, however, interesting to note that F_g computed from their experimental data corresponds roughly to the case $F_g = 0·1$ considered in figure 8. The range of particle sizes used in the experiment corresponded to a range of β^{-1} where the $\tau_{22}^{(p)}$ curve shows a significant decline. Snyder & Lumley, however, seem to suggest that $\tau_{22}^{(p)}$ would continue to fall as the particle size was increased, asymptotically approaching zero. This is not, however, consistent with this formulation. Beyond the minimum the particle time scale $\tau_{ij}^{(p)}$ increases monotonically, asymptotically approaching the particle relaxation time. In this instance, the particle is totally insensitive to fluid velocity fluctuations and the particle motion is dynamically equivalent to that of a particle moving in a quiescent viscous fluid.

4. Summary and conclusions

The analysis presented above gives a technique for calculating the effect of both crossing trajectories and particle inertia on the dispersion of particles in isotropic, homogeneous and stationary turbulence when the particle–fluid drag is assumed linear and in the presence of gravity or any other constant external force. The results indicate that, in the absence of gravity, the asymptotic particle diffusion coefficient is in general greater than that for the fluid.† Only when gravity and other external forces are imposed can this effect be reversed, the effect of gravity on crossing trajectories becoming significant when the particle settling velocity is greater than the turbulent intensity.

† This result is consistent with a very recent analysis of Pismen & Nir (1978) based on Corrsin's hypothesis.

We mention briefly that there is experimental evidence for an effective long-time particle diffusion coefficient greater than that of the fluid, the effect increasing with particle relaxation time. We specifically refer to the work of Goldschmidt & Householder (1969) and Lilly (1973) in jets and pipes. It is believed here, however, that these results must be viewed with some caution because of the way results were obtained, being based on measurements of particle concentration gradients and on the assumption that a gradient-transport model was a valid description. Although such an assumption may be valid for the type of system we have been discussing, it is not obvious that it will be valid in turbulent shear flow (Corrsin 1975), and the results may only be meaningful in the way the measurements have been performed.

Though we have not done so here, for a particle Reynolds number less than unity the analysis may be readily extended to include the extra forces generated by the particle's acceleration. Tchen has shown that (1.2) is still valid. We recognize that the α_{ij} appearing in the equation

$$\langle u_i(0,0)\, u_j^{(2)}(t)\rangle = \int d\mathbf{k}\, \tilde{Q}_{ij}(k,t) \exp\left(-k_i k_j \alpha_{ij}(t,\beta)\right)$$

[equation (2.21)] is simply half the particle mean-square displacement calculated on the basis of $Q_{ij}(0,\xi)$ for $U_{ij}^{(p)}(\xi)$ and a linear drag law. It is found that this expression is still valid when the extra 'acceleration' forces are included: α_{ij} is still based on $Q_{ij}(0,\xi)$, but is clearly different from its value in (2.22).

In conclusion we emphasize that the situation that has been analysed is to some extent an idealized one. Isotropic turbulence and linear drag were assumed strictly for mathematical simplicity though plausible conjectures were made as to the reality of the latter. It would be advantageous to perform a numerical simulation of particle motion similar to Kraichnan's calculations for fluid-point dispersion, and to compare the results with those of this theory.

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